Pizza and Problems

Fall 2008

Assigned on: October 31, 2008 Due on: October 31, 2008

PROBLEM 1 Find all pairs of real numbers (x, y) satisfying the system of equations

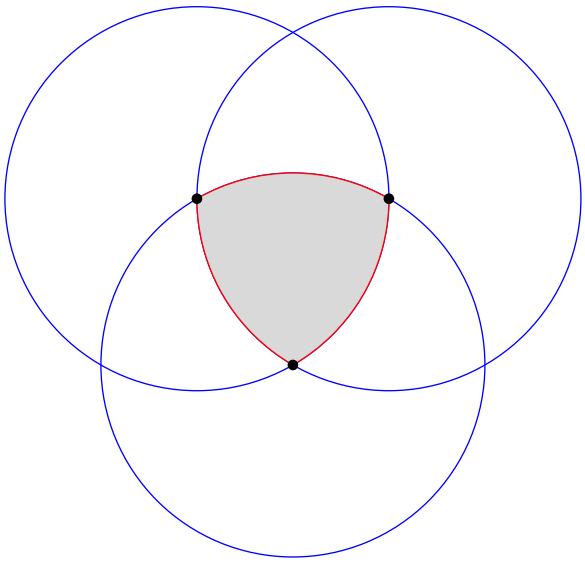
$$\frac{1}{x} + \frac{1}{2y} = (x^2 + 3y^2)(3x^2 + y^2)$$
$$\frac{1}{x} - \frac{1}{2y} = 2(y^4 - x^4)$$

PROBLEM 2 Determine all integers $n > 1$ such that form

$$\frac{2^n + 1}{n^2}$$

is an integer.

PROBLEM 3 In *Problematical Recreations*, No. 7, a series of puzzle booklets once issued annually by Litton Industries in Beverly Hills, California, the following problem appeared. A man places his beer glass on the bar three times to produce the set of triple rings shown in the figure that follows. He does it carefully, so that each circle passes through the center of the other two.



The bartender thinks the area of mutual overlap (shaded) is less than one-fourth of the area of a circle, but the customer says it is more than one-fourth. Who is right?

The solution can be obtained the hard way by finding the area of an equilateral triangle inscribed in he shaded section and then adding the areas of the three segments of the circles

on each side of the triangle. A reader of this column, Tad Dunne of Willdale in Ontario, sent me a beautiful graphical "look and see" solution that involves no geometrical formulas and almost no arithmetic, although it does make use of a repeating wallpaper pattern. Can you rediscover it?

PROBLEM 4 If a penny rolls around another penny without slipping, how many times will it rotate in making one revolution?

PROBLEM 5 The integers 1, 3, 8, and 120 form a set with a remarkable property: the product of any two integers is one less than a perfect square. Find a fifth number that can be added to the set without destroying this property.

PROBLEM 6 Find a number base other than 10 in which 121 is a perfect square.

PROBLEM 7 (Water and Wine) There are two beakers, one containing water, the other wine. A certain amount of water is transferred to the wine, then the same amount of the mixture is transferred back to the water. Is there now more water in the wine than there is wine in the water?

PROBLEM 8 Let a, b, c, and d be integers with $a > b > c > d > 0$. Suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

PROBLEM 9 Three times Dick's age plus Tom's age equals twice Harry's age. Double the cube of Harry's age is equal to three times the cube of Dick's age added to the cube of Tom's age. There respective ages are relatively prime to each other. What is the sum of the squares of their ages?

1 Wiki Page

Our wiki page for Pizza and Problems is located at the following URL:

http://msenux.redwoods.edu/wiki/index.php/Pizza_and_Problems

If interested in editing solutions on this page, you need an account. If you wish an account, post an email to david-arnold@redwoods.edu that includes a username and password which you wish to use to log into the wiki.[1]

[1] Some problems on this page are extracted from a wonderful text by Martin Gardner, *The Collosal Book of Short Puzzles and Problems*, Norton Publishing. Other problems come from past Putnam and Math Olympiad examinations, and some come from Contest Problem Books published by the American Mathematical Society.