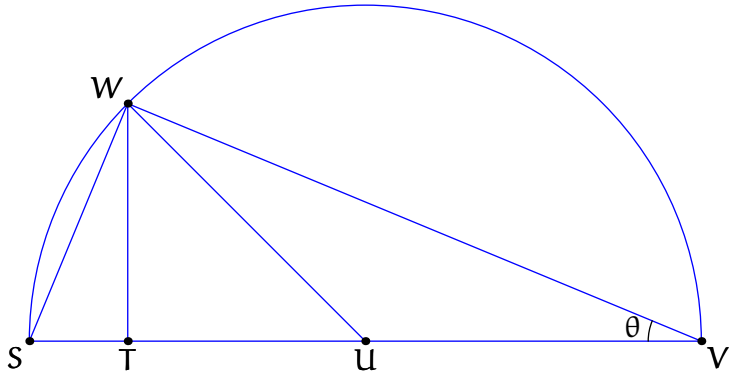
Pizza and Problems

Spring 2008

Assigned on: March 7, 2008 Due on: March 7, 2008

PROBLEM 1 The image that follows is a semicircle of radius 1 centered at the point U. A point W is selected on the semicircle and segment WT is drawn perpendicular to the diameter ST.



Use the figure to prove the following two identities:

(a) $\sin 2\theta = 2\sin\theta\cos\theta$

(b) $\cos 2\theta = 2\cos^2\theta - 1$

PROBLEM 2 The equations of the lines M and N are $y = mx$ and $y = nx$, respectively. Suppose that M makes twice as large an angle with the horizontal (measured counterclockwise from the positive x-axis) as does N, and that M has 4 times the slope of N. If M is not horizontal, find the value of mn.

PROBLEM 3 Given that $i^2 = -1$, for how many integers n is $(n+i)^4$ an integer?

PROBLEM 4 In a triangle with sides of lengths a, b, and c,

$$(a+b+c)(a+b-c) = 3ab.$$

Find the measure of the angle opposite the side of length c.

PROBLEM 5 The triangle ABC is a right triangle with $\angle BCA = 90^\circ$. Median CM is perpendicular to median BN, and side $BC = s$. Find the length of BN in terms of s.

PROBLEM 6 Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n)+1)$ if and only if $n = 1$. [Editor's note: one must assume f is nonconstant.]

PROBLEM 7 In the figure that follows, we have a circle centered at O and passing through the points A, D, and C. Further, arc ABC is a portion of a circle with center $D(0, -1)$. The shaded crescent-shaped region is called a lune. Verify the following result, which was discovered by the Greek mathematician Hippocrates of Chios approximately 2500 years ago: *The area of the lune is equal to the area of the square* OCED.